# Nonperturbative approach to a simple model with ultravioletly divergent eigenenergies in perturbation theory

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## Abstract

We study a simple quantized model for which perturbation theory gives ultravioletly divergent results. We show that when the eigen-solution problem of the Hamiltonian of the model is treated nonperturbatively, it is possible for eigenenergies of the Hamiltonian to be finite.

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## I. INTRODUCTION

Models in quantum field theories usually have the problem of ultraviolet divergence in the framework of perturbation theory (see, e.g., [1–3]). For renormalizable models, ultraviolet divergence can be removed by renormalization techniques, but at the cost that renormalized masses of particles can not be explained in the framework of the theory. On the other hand, non-renormalizable models are usually rejected, since for them it is not clear at the present stage how to obtain finite results for high order corrections in perturbation theory. However, rigorously to say, the break-down of perturbative treatment to a model does not necessarily mean that the model cannot give meaningful results when it is treated nonperturbatively. The point here is that nonperturbative approach to behaviors of states of ultravioletly divergent models is generally quite difficult. In particular, it is still not clear whether the Hamiltonian of a model suffering ultraviolet divergence in the framework of perturbation theory could have eigenstates with finite energies. This problem is of interest, since, if the answer is positive, then, the ultraviolet divergence may be avoidable in nonperturbative treatment and finite energies of ground states may be associated with masses of particles observed experimentally.

In this paper we will show that the answer is indeed positive. For this purpose, we study a simple quantized model, which gives ultravioletly divergent results in the framework of perturbation theory. The model has an interaction structure similar to that of QED in the number representation, so that most of the arguments given to it can be extended to the case of QED. (In this paper, by interaction structure in a representation, we mean the way in which basis states of the representation are coupled by the interaction term of the Hamiltonian, that is, the structure of the non-zero off-diagonal elements of the Hamiltonian matrix in the representation.) For this reason, whether the simple model has an appropriate form in configuration space does not matter here.

Similar to QED, the Hilbert space of the simple model studied in this paper is infinite even when the momentum space is cut off and discretized. For the infinite Hilbert space,

not all the theorems for eigen-solutions of Hamiltonians in finite Hilbert spaces hold. For example, eigenstates of the Hamiltonian of the simple model do not span the whole infinite Hilbert space. In particular, we find that eigenstates and eigenenergies of the Hamiltonian can be expressed in terms of themselves, and, as a result, the eigenenergies can remain finite when the cut-off of momentum is taken off, even if second order corrections to the eigenenergies in perturbation theory is ultravioletly divergent.

Concretely, this paper is organized in the following way. In section II, we introduce the quantized model, discuss the structure of basis states and the structure of the Hamiltonian matrix in the basis states. In section III, we truncate the infinite Hilbert space of the model, so that obtain a series of finite Hilbert spaces, the limit of which gives the infinite Hilbert space. For each truncated finite Hilbert space, we construct another set of basis states by making use of energy eigenstates of another finite Hilbert space. The Hamiltonian matrix in the representation of the new set of basis states is quite simple and can be diagonalized easily. Section IV is devoted to discussions for eigenstates and eigenenergies of the Hamiltonian of the model in the infinite Hilbert space. We show that it is possible for the eigenenergies to be finite, even if perturbative treatment gives divergent results. Conclusions and discussions are given in section V.

## II. A SIMPLE QUANTIZED MODEL

Since realistic models, such as the standard model, are complicated, in this paper, as a first step to the method that is to be developed for nonperturbative approach to eigenstates and eigenenergies of Hamiltonians in quantum field theories, we choose a model as simple as possible to study. Such a model should have an interaction structure similar to that of QED and may have ultraviolet divergence, so that the method developed in this paper for the model can be extended to treat realistic models, such as QED and the standard model. Since here we are interested in properties of eigen-solutions of Hamiltonians only, we are not to start from a classical Lagrangian expressed in configuration space, but start from a

Hamiltonian expressed in terms of creation and annihilation operators for free fields.

The simplest model satisfying the above requirements is composed of a quantized fermion field and a quantized boson field in 1-dimensional momentum space. Denoting the creation and annihilation operators for a free fermion field with momentum $p$ and for a free boson field with momentum $k$ as $b^\dagger(p)$, $b(p)$, and $a^\dagger(k)$, a(k), respectively, the Hamiltonian of the model is taken as

$$H = H_f + H_b + H_I \tag{1}$$

where

$$H_f = \int p_0 b^\dagger(p) b(p) dp \tag{2}$$

$$H_b = \int k_0 a^\dagger(k) a(k) dk \tag{3}$$

$$H_I = \int \Big( V(p_1, p_2, k) b^\dagger(p_2) b(p_1) a^\dagger(k) + h.c. \Big) \, \delta(p_1 - p_2 - k) dp_1 dp_2 dk \tag{4}$$

with $p_0 = |p|$ and $k_0 = |k|$. The operators $b^\dagger(p)$ and $b(p')$ satisfy the usual anticommutation relations and $a^\dagger(k)$ and $a(k')$ satisfy the usual commutation relations. Here $\hbar$ and $c$ are taken to be unit, $\hbar = c = 1$. The interaction structure of this Hamiltonian in the number representation, given by the expression of $H_I$ in Eq. (4), is clearly similar to (although simpler than) that of QED.

For the sake of convenience in discussing properties of the Hamiltonian $H$, we discretize the momentum space and take a cut-off $\Lambda$, that is, we take

$$p = p_i = i \cdot \Delta p - \Lambda, \qquad\qquad k = k_j = j \cdot \Delta p - \Lambda, \tag{5}$$

where $i, j = 0, 1, \ldots, N$ with $N = 2\Lambda/\Delta p$. The Hamiltonian of the model expressed in terms of summations over $p = p_i$ and $k = k_j$ is

$$H(\Lambda) = H_f(\Lambda) + H_b(\Lambda) + H_I(\Lambda), \tag{6}$$

where

$$H_f(\Lambda) = \sum_p p_0 b^\dagger(p) b(p) \tag{7}$$

$$H_b(\Lambda) = \sum_k k_0 a^\dagger(k) a(k) \tag{8}$$

$$H_I(\Lambda) = \sum_{p_1,p_2} \Big( V(p_1,p_2) b^\dagger(p_2) b(p_1) a^\dagger(p_1 - p_2) + h.c. \Big) \,. \tag{9}$$

In the limit of $\Delta p \to 0$ and $\Lambda \to \infty$, $H(\Lambda)$ becomes the $H$ in Eq. (1). In this paper we assume that with finite cut-off $\Lambda$ the model is free from divergence.

Since $H_I(\Lambda)a^\dagger(k)|0\rangle = 0$, where $|0\rangle$ is the vacuum state, states $a^\dagger(k)|0\rangle$, denoted by $|s_k\rangle$ after normalization, are eigenstates of the Hamiltonian $H(\Lambda)$. We are not interested in this kind of trivial eigenstates here. What we are interested in are fermion-type eigenstates. The simplest fermion-type eigenstates of the Hamiltonian $H(\Lambda)$ are in the Hilbert space spanned by the state $b^\dagger(p)|0\rangle$, denoted by $|f_p\rangle$ after normalization, and all the states that can be coupled to $|f_p\rangle$ by $H_I^m(\Lambda)$, the product of $m$ $H_I(\Lambda)$, with $m = 0, 1, 2, \ldots$. It is this Hilbert space that we are to study in this paper, which will be denoted by $L_\infty(p, \Lambda)$.

Concretely to say, basis states of the Hilbert space $L_\infty(p, \Lambda)$ can be taken as

$$|f_{p-k_1-\ldots-k_m} s_{k_1} \ldots s_{k_m}\rangle \equiv N_{p,k_1,\ldots,k_m} b^\dagger(p - k_1 - \ldots - k_m) a^\dagger(k_1) \ldots a^\dagger(k_m)|0\rangle \tag{10}$$

for $m = 0, 1, 2, \ldots$ (the case for $m = 0$ is just $|f_p\rangle$), where $N_{p,k_1,\ldots,k_m}$ are normalization coefficients. Noticing that the basis states $|f_{p-k_1-\ldots-k_m} s_{k_1} \ldots s_{k_m}\rangle$ for all possible $k_1, \ldots, k_m$ can be coupled to the state $|f_p\rangle$ by $H_I^m(\Lambda)$, we will denote the set of them by $\{H_I^m(\Lambda)|f_p\rangle\}$ in what follows. Then, the basis states of the infinite Hilbert space $L_\infty(p, \Lambda)$ in Eq. (10) are elements of the following sets

$$\{H_I^0(\Lambda)|f_p\rangle = |f_p\rangle\}, \{H_I(\Lambda)|f_p\rangle\}, \{H_I^2(\Lambda)|f_p\rangle\}, \{H_I^3(\Lambda)|f_p\rangle\}, \ldots . \tag{11}$$

In some cases in the following sections, for brevity, instead of the expression in Eq. (10), we use $|\xi_{i_m}(m, p, \Lambda)\rangle$ to denote basis states in the set $\{H_I^m(\Lambda)|f_p\rangle\}$, i.e., we use $i_m$ to denote $(k_1, \ldots, k_m)$. For example, $|\xi_{i_0}(0, p, \Lambda)\rangle$ indicates $|f_p\rangle$ with $i_0 = 1$, $|\xi_{i_1}(1, p, \Lambda)\rangle$ indicates $|f_{p-k_1} s_{k_1}\rangle$ with $i_1 = k_1$, and so on.

The interaction structure of the Hamiltonian $H(\Lambda)$ in the basis states in Eq. (10), i.e., the structure of the non-zero off-diagonal elements of the Hamiltonian matrix in the basis

states, has an interesting tree structure: The basis state $|f_p\rangle$ is coupled to basis states in the set $\{H_I(\Lambda)|f_p\rangle\}$ only; basis states in the set $\{H_I(\Lambda)|f_p\rangle\}$ are coupled to basis states in the sets $\{|f_p\rangle\}$ and $\{H_I^2(\Lambda)|f_p\rangle\}$ only; $\ldots$; basis states in the set $\{H_I^m(\Lambda)|f_p\rangle\}$ are coupled to basis states in the sets $\{H_I^{m-1}(\Lambda)|f_p\rangle\}$ and $\{H_I^{m+1}(\Lambda)|f_p\rangle\}$ only; $\ldots$. It is easy to show that, with this structure of the Hamiltonian matrix, when the coupling strength $V(p_1, p_2)$ in Eq. (9) is strong enough, second order corrections to the eigenenergies of the Hamiltonian $H(\Lambda)$ are ultravioletly divergent in perturbation theory when the cut-off $\Lambda$ approaches to infinity.

## III. TRUNCATED HILBERT SPACE AND $\psi_S$ REPRESENTATION

The Hilbert space $L_\infty(p, \Lambda)$ spanned by the basis states in the sets given in (11) is infinite, although the momentum space has been cut off by $\Lambda$. Since the problem of eigen-solutions of a Hamiltonian in an infinite Hilbert space is more difficult than that in a finite Hilbert space and many theorems in the latter case are invalid in the former case, in this section we truncate the Hilbert space to a series of finite ones and discuss properties of the eigenstates of the Hamiltonian $H(\Lambda)$ in the truncated finite Hilbert spaces. Then, in the next section, we discuss what we could have when the truncated finite Hilbert spaces resume the infinite Hilbert space $L_\infty(p, \Lambda)$.

### A. Truncated finite Hilbert spaces

A truncated finite Hilbert space, denoted by $L_n(p, \Lambda)$, is spanned by basis states in a set $A(n, p, \Lambda)$ defined by

$$A(n, p, \Lambda) = \{|f_p\rangle\} \bigcup \{H_I(\Lambda)|f_p\rangle\} \bigcup \ldots \bigcup \{H_I^n(\Lambda)|f_p\rangle\}. \tag{12}$$

When $n$ goes to infinity, the truncated Hilbert space $L_n(p, \Lambda)$ will become the infinite Hilbert space $L_\infty(p, \Lambda)$.

Normalized eigenstates and the corresponding eigenenergies of the Hamiltonian $H(\Lambda)$ in the truncated finite Hilbert space $L_n(p,\Lambda)$ are denoted by $|\psi_{\alpha_n}(n,p,\Lambda)\rangle$ and $E_{\alpha_n}(n,p,\Lambda)$, respectively,

$$H(\Lambda)|\psi_{\alpha_n}(n,p,\Lambda)\rangle = E_{\alpha_n}(n,p,\Lambda)|\psi_{\alpha_n}(n,p,\Lambda)\rangle. \tag{13}$$

These states $|\psi_{\alpha_n}(n,p,\Lambda)\rangle$ also span the Hilbert space $L_n(p,\Lambda)$ and can be expanded in the basis states in the set $A(n,p,\Lambda)$,

$$|\psi_{\alpha_n}(n,p,\Lambda)\rangle = \sum_{m=0}^{n}\sum_{i_m} C_{\alpha_n,i_m}(n,m,p,\Lambda)|\xi_{i_m}(m,p,\Lambda)\rangle, \tag{14}$$

where $C_{\alpha_n,i_m}(n,m,p,\Lambda)$ are expanding coefficients.

The set $A(n,p,\Lambda)$ has an interesting structure: It can be expressed by making use of the sets $A(n-1,p-k,\Lambda)$. To see this, first note that the product of two basis states $|s_k\rangle$ and $|f_{p-k-k_1-\ldots-k_m}s_{k_1}\ldots s_{k_m}\rangle$ is

$$|s_k\rangle\cdot|f_{p-k-k_1-\ldots-k_m}s_{k_1}\ldots s_{k_m}\rangle = |f_{p-k-k_1-\ldots-k_m}s_{k_1}\ldots s_{k_m}s_k\rangle. \tag{15}$$

From Eq. (10), it is easy to see that each basis state in the set $\{H_I^m(\Lambda)|f_p\rangle\}$ with $m\geq 1$ can be expressed as the product of a basis state $|s_k\rangle$ and a basis state in the set $\{H_I^{m-1}(\Lambda)|f_{p-k}\rangle\}$. Then, denoting the set of the product of the basis states $|s_k\rangle$ and the basis states in the set $A(n-1,p-k,\Lambda)$ for all possible $k$ as $B(n-1,p,k,\Lambda)$,

$$B(n-1,p,k,\Lambda) = \{|s_k\rangle\cdot|\xi\rangle : \quad \text{for } |\xi\rangle\in A(n-1,p-k,\Lambda) \;\text{ and all possible } k\}, \tag{16}$$

the set $A(n,p,\Lambda)$ can be reexpressed as

$$A(n,p,\Lambda) = \{|f_p\rangle\}\bigcup_k B(n-1,p,k,\Lambda). \tag{17}$$

In the next subsection, we show that this structure of the set $A(n,p,\Lambda)$ enables us to construct another set of basis states, in which the Hamiltonian matrix can be diagonalized explicitly.

### B. $\psi_s$-representation in truncated finite Hilbert spaces

Making use of the definition of product of basis states given in Eq. (15) and the expansion of eigenstates in Eq. (14), it is easy to get $|s_k\rangle \cdot |\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$, the product of a state $|s_k\rangle$ and an eigenstate $|\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$ in the truncated Hilbert space $L_{n-1}(p-k, \Lambda)$, which will be denoted by $|s_k\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$,

$$|s_k\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle = \sum_{m=0}^{n-1} \sum_{i_m} C_{\alpha_{n-1}, i_m}(n-1, m, p-k, \Lambda)|s_k\rangle \cdot |\xi_{i_m}(m, p-k, \Lambda)\rangle. \quad (18)$$

These states $|s_k\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$ span the same Hilbert space as the states in the set $B(n-1, p, k, \Lambda)$ in Eq. (16). Therefore, these states together with the state $|f_p\rangle$ also span the truncated Hilbert space $L_n(p, \Lambda)$.

Now we show that in the limit of $\Delta p \to 0$, the states $|s_k\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$ are orthogonal to each other and together with $|f_p\rangle$ form another set of orthogonal basis states for the truncated Hilbert space $L_n(p, \Lambda)$. To prove this, we rewrite $|\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$ in the form

$$|\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle = \sum_{m=0}^{n-1} |h_{\alpha_{n-1}}(n-1, m, p-k, \Lambda)\rangle, \quad (19)$$

where

$$|h_{\alpha_{n-1}}(n-1, m, p-k, \Lambda)\rangle = \sum_{i_m} C_{\alpha_{n-1}, i_m}(n-1, m, p-k, \Lambda)|\xi_{i_m}(m, p-k, \Lambda)\rangle, \quad (20)$$

for example,

$$|h_{\alpha_{n-1}}(n-1, 0, p-k, \Lambda)\rangle = C_{\alpha_{n-1}, 1}(n-1, 0, p-k, \Lambda)|f_{p-k}\rangle \quad (21)$$

$$|h_{\alpha_{n-1}}(n-1, 1, p-k, \Lambda)\rangle = \sum_{k_1} C_{\alpha_{n-1}, k_1}(n-1, 1, p-k, \Lambda)|f_{p-k-k_1} s_{k_1}\rangle. \quad (22)$$

The states $|h_{\alpha_{n-1}}(n-1, m, p-k, \Lambda)\rangle$ are projections of the state $|\psi_{\alpha_{n-1}}(n-1, p-k, \Lambda)\rangle$ in the subspaces spanned by states in the sets $\{H_I^m(\Lambda)|f_{p-k}\rangle\}$, respectively, therefore, $|h_{\alpha_{n-1}}(n-1, m, p-k, \Lambda)\rangle$ with different $m$ are orthogonal to each other.

Making use of Eq. (19), we have

$$\langle s_{k'}\psi_{\alpha'_{n-1}}(n-1,p-k',\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle = \sum_{m=0}^{n-1} I_m, \tag{23}$$

where

$$I_m = \langle s_{k'}h_{\alpha'_{n-1}}(n-1,m,p-k',\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle \tag{24}$$

with $|s_{k'}h_{\alpha'_{n-1}}(n-1,m,p-k',\Lambda)\rangle = |s_{k'}\rangle \cdot |h_{\alpha'_{n-1}}(n-1,m,p-k',\Lambda)\rangle$ defined in the same way as $|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle$ in Eq. (18). It is easy to verify that

$$I_0 = C^*_{\alpha'_{n-1},1}(n-1,0,p-k',\Lambda)C_{\alpha_{n-1},1}(n-1,0,p-k,\Lambda)\delta_{kk'} \tag{25}$$

$$\begin{aligned} I_1 &= C^*_{\alpha'_{n-1},k}(n-1,1,p-k',\Lambda)C_{\alpha_{n-1},k'}(n-1,1,p-k,\Lambda) \\ &\quad + \sum_{k_1} C^*_{\alpha'_{n-1},k_1}(n-1,1,p-k',\Lambda)C_{\alpha_{n-1},k_1}(n-1,1,p-k,\Lambda)\delta_{kk'}. \end{aligned} \tag{26}$$

Here we assume that in the limit of $\Delta p \to 0$, there is no singularity in the coefficients $C_{\alpha_{n-1},i_m}(n-1,m,p-k,\Lambda)$ for fixed $n-1$ and $m$, i.e., $C_{\alpha_{n-1},i_m}(n-1,m,p-k,\Lambda)$ is a smooth function of $i_m$. Then, since

$$\sum_{k_1} |C_{\alpha_{n-1},k_1}(n-1,1,p-k,\Lambda)|^2 < 1, \tag{27}$$

when the interval $\Delta p$ for discretizing the momentum $k_1$ goes to zero, the value of $C^*_{\alpha'_{n-1},k}(n-1,1,p-k',\Lambda)C_{\alpha_{n-1},k'}(n-1,1,p-k,\Lambda)$ should approach to zero, and Eq. (26) gives

$$\lim_{\Delta p \to 0} I_1 = \sum_{k_1} C^*_{\alpha'_{n-1},k_1}(n-1,1,p-k',\Lambda)C_{\alpha_{n-1},k_1}(n-1,1,p-k,\Lambda)\delta_{kk'}. \tag{28}$$

Similar results can also be obtained for the other $I_m$, and finally we have

$$\begin{aligned} &\lim_{\Delta p \to 0}\langle s_{k'}\psi_{\alpha'_{n-1}}(n-1,p-k',\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle \\ &= \sum_{m=0}^{n-1}\sum_{i_m} C^*_{\alpha'_{n-1},i_m}(n-1,m,p-k',\Lambda)C_{\alpha_{n-1},i_m}(n-1,m,p-k,\Lambda)\delta_{kk'} = \delta_{\alpha_{n-1}\alpha'_{n-1}}\delta_{kk'}. \end{aligned} \tag{29}$$

That is to say, the states $|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle$ are normalized and orthogonal to each other. Therefore, these states and the state $|f_p\rangle$ form another set of normalized orthogonal basis states for the truncated Hilbert space $L_n(p,\Lambda)$. The representation given by this set of basis states in the limit of $\Delta p \to 0$, will be termed $\psi_s$-*representation* in what follows.

The Hamiltonian $H(\Lambda)$ has a quite simple matrix form in the $\psi_s$-representation of the Hilbert space $L_n(p,\Lambda)$. In fact, similar to Eq. (29), one can verify that

$$\lim_{\Delta p\to 0}\langle s_{k'}\psi_{\alpha'_{n-1}}(n-1,p-k',\Lambda)|H(\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle$$
$$=\langle s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)|H(\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle\delta_{\alpha_{n-1}\alpha'_{n-1}}\delta_{kk'}. \tag{30}$$

Therefore, the non-zero off-diagonal elements of the Hamiltonian matrix in the $\psi_s$-representation are those connecting the state $|f_p\rangle$ and the states $|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle$ only. Diagonalization of the Hamiltonian matrix with such a simple structure is quite easy, which gives

$$E_{\alpha_n}(n,p,\Lambda)-\langle f_p|H(\Lambda)|f_p\rangle$$
$$=\sum_{k,\alpha_{n-1}}\frac{\Big|\langle f_p|H(\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle\Big|^2}{E_{\alpha_n}(n,p,\Lambda)-\langle s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)|H(\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle} \tag{31}$$

and

$$|\psi_{\alpha_n}(n,p,\Lambda)\rangle=D_{\alpha_n}(n,p,\Lambda)|f_p\rangle+\sum_{k,\alpha_{n-1}}D_{\alpha_n,k\alpha_{n-1}}(n,p,\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle, \tag{32}$$

where

$$D_{\alpha_n,k\alpha_{n-1}}(n,p,\Lambda)=\frac{\langle s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)|H(\Lambda)|f_p\rangle\cdot D_{\alpha_n}(n,p,\Lambda)}{E_{\alpha_n}(n,p,\Lambda)-\langle s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)|H(\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle} \tag{33}$$

and

$$D_{\alpha_n}(n,p,\Lambda)$$
$$=\left[1+\sum_{k,\alpha_{n-1}}\frac{|\langle s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)|H(\Lambda)|f_p\rangle|^2}{\Big(E_{\alpha_n}(n,p,\Lambda)-\langle s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)|H(\Lambda)|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle\Big)^2}\right]^{-\frac{1}{2}}. \tag{34}$$

## IV. EIGENENERGIES AND EIGENSTATES IN INFINITE HILBERT SPACE

When $n$ goes to infinity, the truncated Hilbert space $L_n(p,\Lambda)$ discussed in the previous section will become the infinite Hilbert space $L_\infty(p,\Lambda)$. However, eigenstates of the Hamiltonian $H(\Lambda)$ in the infinite Hilbert space cannot be obtained by simply letting the index $n$ go

to infinity in the eigenstates $|\psi_{\alpha_n}(n,p,\Lambda)\rangle$ of the truncated Hilbert space $L_n(p,\Lambda)$. In fact, when $n$ becomes $n+1$, the number of eigenstates in the truncated Hilbert space will become $N$ times larger ($N$ is the number of discretized momenta); and despite of how large $n$ is, there exist eigenstates $|\psi_{\alpha_n}(n,p,\Lambda)\rangle$ for which the values of $|\langle h_{\alpha_n}(n,n,p,\Lambda)|\psi_{\alpha_n}(n,p,\Lambda)\rangle|^2$ are not small, i.e., there exist eigenstates whose projections in the subspace spanned by states in the set $\{H_I^n(\Lambda)|f_p\rangle\}$ are not small. Therefore, the definition for eigenstates of the Hamiltonian $H(\Lambda)$ in the infinite Hilbert space $L_\infty(p,\Lambda)$ should be treated more carefully. In subsection IV A, we give the definition and discuss some properties of the eigenstates defined by it. In subsection IV B, we show that eigenstates of the Hamiltonian $H(\Lambda)$ in the infinite Hilbert space $L_\infty(p,\Lambda)$ can be expressed in terms of themselves, and the corresponding eigenenergies can remain finite when the cut-off $\Lambda$ is taken off, even if they are ultravioletly divergent in perturbation theory. Subsection IV C is devoted to a brief discussion for evolution of states with time in the infinite Hilbert space.

### A. Energy eigenstates in infinite Hilbert space

A normalized eigenstate of the Hamiltonian $H(\Lambda)$ in the infinite Hilbert space $L_\infty(p,\Lambda)$, denoted by $|\phi_\beta(p,\Lambda)\rangle$, with eigenenergy $E_\beta(p,\Lambda)$ is defined in the following way: For each positive number $\epsilon$, there exists a number $N(\epsilon)$ such that for each $n$ not smaller than $N(\epsilon)$, there exists an eigenstate $|\psi_\beta(n,p,\Lambda)\rangle$ of the Hamiltonian $H(\Lambda)$ with eigenenergy $E_\beta(n,p,\Lambda)$ in the truncated Hilbert space $L_n(p,\Lambda)$ satisfying

$$1-|\langle\psi_\beta(n,p,\Lambda)|\phi_\beta(p,\Lambda)\rangle|^2<\epsilon \tag{35}$$

and

$$|E_\beta(p,\Lambda)-E_\beta(n,p,\Lambda)|<\epsilon. \tag{36}$$

Then, each eigenstate $|\phi_\beta(p,\Lambda)\rangle$ is the limit of a series of states in the truncated Hilbert spaces $L_n(p,\Lambda)$,

$$|\phi_\beta(p,\Lambda)\rangle = \lim_{n\to\infty} |\psi_\beta(n,p,\Lambda)\rangle. \tag{37}$$

A property of an eigenstate $|\phi_\beta(p,\Lambda)\rangle$ is that its projection is less than $\epsilon$ in the subspace of the infinite Hilbert space spanned by states in the sets $\{H_I^{N(\epsilon)+1}\}$, $\{H_I^{N(\epsilon)+2}\}$, .... In fact, substituting Eq. (35) into the normalization condition

$$\sum_{\alpha_{N(\epsilon)}} |\langle\psi_{\alpha_{N(\epsilon)}}(N(\epsilon),p,\Lambda)|\phi_\beta(p,\Lambda)\rangle|^2 + \sum_{m=N(\epsilon)+1}^{\infty} \sum_{i_m} |\langle\xi_{i_m}(m,p,\Lambda)|\phi_\beta(p,\Lambda)\rangle|^2 = 1, \tag{38}$$

we have

$$\sum_{m=N(\epsilon)+1}^{\infty} \sum_{i_m} |\langle\xi_{i_m}(m,p,\Lambda)|\phi_\beta(p,\Lambda)\rangle|^2 < \epsilon. \tag{39}$$

Therefore, the projection of each of the eigenstates $|\phi_\beta(p,\Lambda)\rangle$ in the subspace spanned by states in the set $\{H_I^n(\Lambda)|f_p\rangle\}$ approaches to zero when $n \to \infty$. Due to this property of the eigenstates $|\phi_\beta(p,\Lambda)\rangle$, one can see that for large $n$ and small $\epsilon$, the number of the states $|\psi_\beta(n,p,\Lambda)\rangle$, which are close to the eigenstates $|\phi_\beta(p,\Lambda)\rangle$, must be much smaller than the total number of the states $|\psi_{\alpha_n}(n,p,\Lambda)\rangle$ in the truncated Hilbert space $L_n(p,\Lambda)$. As a result, the eigenstates $|\phi_\beta(p,\Lambda)\rangle$ span a small part of the infinite Hilbert space $L_\infty(p,\Lambda)$ only.

### B. Expressions of eigenenergies and eigenstates in $\phi_s$-subspace

As discussed above, when $n \to \infty$, not all the states $|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle$, but a fraction of them, namely, $|s_k\psi_\beta(n-1,p-k,\Lambda)\rangle$ have definite limit

$$\lim_{n\to\infty} |s_k\psi_\beta(n-1,p-k,\Lambda)\rangle = |s_k\phi_\beta(p-k,\Lambda)\rangle. \tag{40}$$

The subspace of the Hilbert space $L_\infty(p,\Lambda)$ spanned by the states

$$|f_p\rangle \quad \text{and} \quad |s_k\phi_\beta(p-k,\Lambda)\rangle \tag{41}$$

for all possible $k$ and $\beta$ will be termed $\phi_s$-*subspace*. Since all the states $|s_k\psi_{\alpha_{n-1}}(n-1,p-k,\Lambda)\rangle$ with $\alpha_{n-1} \neq \beta$ do not have definite limit when $n$ goes to $\infty$, the eigenstates $|\phi_\beta(p,\Lambda)\rangle$ must lie in the $\phi_s$-subspace of the infinite Hilbert space $L_\infty(p,\Lambda)$. Therefore, if one diagonalizes the Hamiltonian $H(\Lambda)$ in the $\phi_s$-subspace, the eigenstates obtained in the $\phi_s$-subspace

must contain all the eigenstates $|\phi_\beta(p,\Lambda)\rangle$ of the whole Hilbert space $L_\infty(p,\Lambda)$. As a result, in order to obtain the eigenstates $|\phi_\beta(p,\Lambda)\rangle$, one does not need to diagonalize the Hamiltonian $H(\Lambda)$ in the whole Hilbert space $L_\infty(p,\Lambda)$, but diagonalization in the $\phi_s$-subspace is enough.

Similar to Eq. (29), one can show that the states in (41) are normalized and orthogonal to each other. Furthermore, the Hamiltonian matrix in the $\phi_s$-subspace has a structure similar to that in the $\psi_s$-representation of the Hilbert space $L_n(p,\Lambda)$, i.e., there is coupling between the state $|f_p\rangle$ and each of the states $|s_k\phi_\beta(p-k,\Lambda)\rangle$, but the coupling between each two of the states $|s_k\phi_\beta(p-k,\Lambda)\rangle$ approaches to zero when $\Delta p \to 0$. Then, similar to Eqs. (31) and (32), in the limit of $\Delta p \to 0$, one can obtain expressions of the eigenstates and eigenenergies of the Hamiltonian $H(\Lambda)$ in the $\phi_s$-subspace. In particular, for eigenenergies and eigenstates of the Hamiltonian $H(\Lambda)$ in the whole Hilbert space $L_\infty(p,\Lambda)$, after simplification we have

$$E_\beta(p,\Lambda) = p_0 + {\sum_{k,\beta'}}' \frac{|\langle f_p|H(\Lambda)|f_{p-k}s_k\rangle|^2 \cdot |d_{\beta'}(p-k,\Lambda)|^2}{E_\beta(p,\Lambda) - k_0 - E_{\beta'}(p-k,\Lambda)} \tag{42}$$

$$|\phi_\beta(p,\Lambda)\rangle = D_\beta(p,\Lambda)|f_p\rangle + {\sum_{k,\beta'}}' D_{\beta,k\beta'}(p,\Lambda)|s_k\phi_{\beta'}(p-k,\Lambda)\rangle, \tag{43}$$

where

$$D_{\beta,k\beta'}(p,\Lambda) = \frac{\langle f_{p-k}s_k|H(\Lambda)|f_p\rangle d_{\beta'}(p-k,\Lambda)}{E_\beta(p,\Lambda) - k_0 - E_{\beta'}(p-k,\Lambda)} D_\beta(p,\Lambda) \tag{44}$$

$$D_\beta(p,\Lambda) = \left[1 + {\sum_{k,\beta'}}' \frac{|\langle f_p|H(\Lambda)|f_{p-k}s_k\rangle|^2 \cdot |d_{\beta'}(p-k,\Lambda)|^2}{(E_\beta(p,\Lambda) - k_0 - E_{\beta'}(p-k,\Lambda))^2}\right]^{-\frac{1}{2}} \tag{45}$$

$$d_{\beta'}(p-k,\Lambda) = \langle \phi_{\beta'}(p-k,\Lambda)|f_{p-k}\rangle \tag{46}$$

and the primes over the summations mean that when $k_0 = 0$ the index $\beta'$ is not equal to $\beta$. Note that the right hand sides of Eqs. (42) and (43) contain the eigenenergies and eigenstates themselves.

Now let us compare the expression of the eigenenergy $E_\beta(p,\Lambda)$ on the right hand side of Eq. (42) and the second order correction for it in perturbation theory, which is

$$E_\beta^{(2)}(p,\Lambda) = {\sum_k}' \frac{|\langle f_p|H_I(\Lambda)|f_{p-k}s_k\rangle|^2}{p_0 - |p-k| - k_0}. \tag{47}$$

The main difference between the right hand side of Eq. (42) and the right hand side of Eq. (47) is that there is a term $|d_{\beta'}(p-k,\Lambda)|^2$ in the numerator of the right hand side of Eq. (42). Equation (46) shows that this term is less than one. Let us consider the case that the summation on the right hand side of Eq. (47) goes to infinity as the cut-off $\Lambda \to \infty$. For the summation on the right hand side of Eq. (42), if the value of $|d_{\beta'}(p-k,\Lambda)|^2$ decreases fast enough with increasing $k_0$, then, it would be possible for it to be convergent in the limit of $\Lambda \to \infty$. That is to say, for a model suffering ultraviolet divergence in perturbation theory, when it is treated nonperturbatively, it is possible for some of, even all of, its eigenstates to have finite eigenenergies.

At last, we would like to mention that the theory of relativity may give additional restrictions to possible physical eigenstates with finite eigenenergies in the limit of $\Lambda \to \infty$. For example, the requirement that $E_\beta(p,\infty)$ is finite does not guarantee that it satisfies the relation

$$E^2_\beta(p,\infty) = E^2_\beta(0,\infty) + p^2. \tag{48}$$

Furthermore, the theory of relativity requires that Lorentzian transformation can transform an eigenstate $|\phi_\beta(p,\infty)\rangle$ to an eigenstate $|\phi_\beta(p',\infty)\rangle$. But, it is not clear if all the eigenstates given in Eq. (43) satisfy this requirement.

### C. Evolution of states in infinite Hilbert space

Finally, let us give a brief discussion for evolution of states in the infinite Hilbert space $L_\infty(p,\lambda)$. In the infinite Hilbert space, the subspace spanned by the eigenstates $|\phi_\beta(p,\Lambda)\rangle$ of the Hamiltonian $H(\Lambda)$, which will be called *eigen-subspace* , play a special role. States in the eigen-subspace can be expanded in the eigenstates $|\phi_\beta(p,\Lambda)\rangle$, therefore, they evolve in the same way as those in a finite Hilbert space, say, for an initial state $|\Phi(t=0)\rangle = |\Phi_0\rangle$,

$$|\Phi(t)\rangle = \sum_{p,\beta} \langle \phi_\beta(p,\Lambda)|\Phi_0\rangle e^{-iE_\beta(p,\Lambda)t} |\phi_\beta(p,\Lambda)\rangle. \tag{49}$$

In the general case, evolution of states is more complicated than that in Eq. (49). For example, consider a state $|\Psi_0\rangle$ lie in a finite subspace $L_n(p,\lambda)$ of the infinite Hilbert space $L_\infty(p,\lambda)$. It can be divided into two parts: one in the eigen-subspace, denoted by $|\Psi_0^{es}\rangle$, the other out side of the eigen-subspace, denoted by $|\Psi_0^{nes}\rangle$. Then, the $|\Psi_0^{es}\rangle$ part of the state will evolve as in Eq. (49). But, the evolution of $|\Psi_0^{nes}\rangle$ is not so clear. As time increases, it is possible for it to spread to subspaces of the Hilbert space spanned by states in the sets $\{H_I^m(\Lambda)|f_p\rangle\}$ for whatever large $m$. In this case, when we are concerned with properties of the state in a finite subspace $L_n(p,\Lambda)$ only, the probability of the state $|\Psi^{nes}(t)\rangle$ in the subspace $L_n(p,\Lambda)$ will become smaller and smaller as $t$ increases, which remind one of properties of dissipative systems. At the present stage, it is not clear if this feature of evolution of states is common to all the states out side of the eigen-subspace.

## V. CONCLUSIONS AND DISCUSSIONS

In this paper, we study a simple quantized model, which has an interaction structure similar to (but simpler than ) that of QED and gives ultravioletly divergent results in the framework of perturbation theory. We show that when the eigen-problem of the Hamiltonian of the model is treated nonperturbatively, it is in fact possible for eigenenergies of the Hamiltonian to be finite. The eigenstates of the Hamiltonian are found to span part of the whole infinite Hilbert space only. Evolution of states in the infinite Hilbert space shows features quite different from that in finite Hilbert spaces.

We expect that the method introduced in this paper is also useful in studying more realistic models, such as the standard model and models including the gravity. In the application of this method to realistic models, masses of particles are to be explained as energies of ground states of interacting fields, therefore, there would be no need to resort to Higgs mechanism to get masses for particles. For the standard model, without introducing masses to free fields, the Hamiltonian of the model can not have any non-zero finite eigenenergy due to the lack of the dimension of mass. But, making use of the method discussed in this

paper, it would be possible for one to get information on ratios of the eigenenergies and properties of the eigenstates of the Hamiltonian of the standard model. In order to obtain non-zero finite eigenenergies without introducing masses to free fields, one must include the gravity. One of the most serious problems one meets when including the gravity in quantum field theories is that the extended theories are generally non-renormalizable. However, as discussed in this paper, it is not impossible for non-renormalizable models to have energy eigenstates with finite energies when treated rigorously. Therefore, besides the string theory (see, e.g., [4,5]), the method introduced in this paper may supply another possible way of overcoming the difficulty of ultraviolet divergence.

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